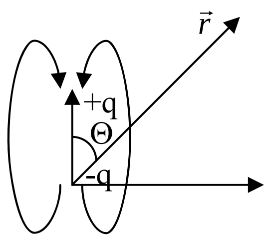
Elektrinis dipolis



Tokios sistemos elektrinis laukas ir potencialas:

$$\varphi = \frac{(\vec{p}\vec{r})}{4\pi\varepsilon_0 r^2} = \frac{p\cos\Theta}{4\pi\varepsilon\, r^2} \qquad \oint_S (\vec{D}d\vec{S}) = 0$$

Prisiminus, kad: $\quad \vec{E} = -\mathrm{grad}\varphi$

$$\mathrm{div}\vec{E} = -\mathrm{divgrad}\varphi = -\Delta\varphi$$

$$\int_V \rho dV = \varepsilon_0 \int_V \mathrm{div}\vec{E}dV = \int_S (\varepsilon\, \vec{E}d\vec{S}) = 0$$

Turim spręst Laplaso lygtį: $\quad \Delta\varphi = 0$

Spręsdami lygtį sužinome, ar gaunasi tas mūsų dipolis.

$\varphi = f(r)F(\Theta\alpha)$ - Laplaso sprendinys kaip dviejų f-jų sandauga

Sprendžiam sferinėj koordinačių sistemoj:

$$\frac{1}{r^2\sin\Theta}\left[\sin\Theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\varphi}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2\varphi}{\partial\alpha^2}\right] = \Delta\varphi = \mathrm{divgrad}\varphi = 0$$

Jei laikom, kad $r \neq 0$, tokioj erdvės daly gaunam:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\varphi}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2\varphi}{\partial\alpha^2} = 0$$

$$F\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + f\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial F}{\partial\Theta}\right) + f\frac{1}{\sin^2\Theta}\frac{\partial^2 F}{\partial\alpha^2} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = C\cdot f; \qquad f \neq 0; \qquad \rightarrow \qquad \text{suprastinam nuos sios funkcijos}$$

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = 2r\frac{df}{dr} + r^2\frac{d^2f}{dr^2} = C\cdot f; \qquad f = A_l\cdot r^l$$

$$A_l\left(2rlr^{l-1} + r^2 l(l-1)r^{l-2}\right) = C\cdot A_l\cdot r^l$$

$C = l(l+1);$ $\qquad$ Tikrine funkcija $\rightarrow f_l = A_l\cdot r^l$

Funkcija yra antros eilės, todėl bus ir 2 sprendiniai. Ieškosim antro sprendinio:

$$[-(l+1)][-(l+1)+1] = l(l+1)$$

$$f_2 = B_l\cdot r^{-(l+1)}$$

Bendras sprendinys: $\quad f_l = A_l r^l + B_l\frac{1}{r^{l+1}}$

Tegu $l = 0, 1, 2, \ldots$

<u>Kai $l = 0$:</u>

$$f_0 = A_0 + \frac{B_0}{r} \Rightarrow F = \mathrm{const} \Rightarrow \varphi = C + \frac{B}{r} \quad \rightarrow \text{Gavosi taskinio kruvio potencialas}$$

<u>Kai l = bet koks:</u>

$$A_l f_l\left(F l(l+1) + \frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial F}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2 F}{\partial\alpha^2}\right) = 0$$

$f_l \neq 0; \qquad \rightarrow \qquad$ is jo suprastinam

$$\underbrace{\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial F}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2 F}{\partial\alpha^2}}_{\text{Sprendžiam šitą lygtį}} + l(l+1)F = 0$$

$$\left(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2\right)F = -l(l+1)F$$

$$\hat{L}_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\hat{L}_z^2 = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)F$$

$F(\Theta,\alpha) = \Theta(\vartheta)\cdot\alpha(\alpha)$ - Šitą sprendinį surašom į tą lygtį

Po apdorojimų gaunam:

$$\frac{1}{\Theta(\vartheta)}\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{d\Theta(\vartheta)}{d\vartheta}\right) + \frac{1}{\alpha}\frac{1}{\sin^2\Theta}\frac{d^2\alpha(\alpha)}{d\alpha^2} + l(l+1) = 0$$

$$\underbrace{\frac{1}{\Theta(\vartheta)}\sin\Theta\frac{d}{d\Theta}\left(\sin\Theta\frac{d\Theta(\vartheta)}{d\vartheta}\right)}_{\text{(priklauso nuo }\Theta\text{)}} + \underbrace{\frac{1}{\alpha(\alpha)}\frac{d^2\alpha(\alpha)}{d\alpha^2}}_{= -m^2} + l(l+1)\sin^2\Theta = 0$$

$$\frac{d^2\alpha(\alpha)}{d\alpha^2} = \text{const}\cdot\alpha(\alpha); \qquad \alpha(\alpha) = e^{\pm i m \alpha}$$

m = 0, 1, 2, 3, ...,

m = -l, -(l-1), 0, l-1, l

$$\frac{d^2\alpha(\alpha)}{d\alpha^2} = -m^2\alpha(\alpha)$$

$$\frac{1}{\Theta(\vartheta)}\sin\Theta\frac{d}{d\Theta}\left(\sin\Theta\frac{d\Theta(\vartheta)}{d\Theta}\right) - m^2 + l(l+1)\sin^2\Theta = 0$$

Atliekam pakeitimą:

$$\begin{bmatrix} x = \cos\Theta & \dfrac{dx}{d\Theta} = -\sin\Theta \\ \dfrac{d\Theta(\vartheta)}{d\Theta} = \dfrac{d\Theta}{dx}\dfrac{dx}{d\Theta} = -\sin\Theta\dfrac{d\Theta(x)}{dx} & \end{bmatrix}$$

Gaunam:

$$\sin\Theta\frac{d}{d\Theta}\left(\sin\Theta\frac{d\Theta(\vartheta)}{d\Theta}\right) + \left[l(l+1)\sin^2\Theta - m^2\right]\Theta(\vartheta) = 0$$

$$-\sin^2\Theta\frac{d}{dx}\left(-\sin^2\Theta\frac{d\Theta(x)}{dx}\right) + \left[l(l+1)\sin^2\Theta - m^2\right]\Theta(x) = 0$$

$$\left[\sin^2\Theta = 1-\cos^2\Theta = 1 - x^2\right]$$

$$\left(1-x^2\right)\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\Theta(x)}{dx}\right] + \left[l(l+1)(1-x^2) - m^2\right]\Theta(x) = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\Theta(x)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\Theta(x) = 0 \longrightarrow \text{Prijungtinė Lažandro lygtis}$$

$P_l^m(x)$ - Lažandro polinomai; jo lygties sprendiniai

$P_1^0 = \cos\Theta$

$P_1^1 = -P_1^{-1} = -\sin\Theta$

$F(\Theta,\alpha)$ - sferinės f-jos

$F(\Theta,\alpha) = Y_l^m(\Theta,\alpha)$

$Y_1^0(\Theta,\alpha) = \sqrt{\frac{3}{4\pi}}\cos\Theta$

$Y_1^{-1}(\Theta,\alpha) = -Y_1^1(\Theta,\alpha) = \sqrt{\frac{3}{8\pi}}\sin\Theta \cdot e^{\pm i\alpha}$

Kai l = 1:

$m = 0 \qquad \varphi_{10} = B_{10} r\cos\Theta$

$m = 1 \qquad \varphi_{11} = B_{11} r\sin\Theta e^{i\alpha}$

$m = -1 \qquad \varphi_{1\text{-}1} = B_{1-1} r\sin\Theta e^{-i\alpha}$

Kai –(l+1) = -2:

$m = 0 \qquad \varphi_{\text{-}20} = B_{-20}\frac{\cos\Theta}{r}$

$m = 1 \qquad \varphi_{\text{-}21} = B_{-21}\frac{\sin\Theta}{r^2}e^{i\alpha}$

$m = -1 \qquad \varphi_{\text{-}2,\text{-}1} = B_{-2-1}\frac{\sin\Theta}{r^2}e^{-i\alpha}$

$\vec{E} = -\mathrm{grad}\varphi_{-20} = \frac{B_{-20}}{r^3}\left(\vec{r}_0 2\cos\Theta + \vec{\Theta}_0\sin\Theta\right)$ - elektrinio dipolio elektrinis laukas

Pereinant prie Dekarto koord. sist.:

$r = r\cos\Theta$

$\varphi_{10} = B_{10}z$ - pastovus laukas, nukreiptas z ašies kryptimi

$E_{10} = -\mathrm{grad}\varphi_{10} = -B_{10}\vec{z}_0$

Herco vektoriai

$$\vec{E} = -\text{grad}\varphi - \frac{\partial A}{\partial t} \qquad \text{div}\vec{A} + \varepsilon_0\mu_0 \frac{\partial \varphi}{\partial t} = 0$$

$$\vec{B} = \text{rot}\vec{A}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

P – elektrinės poliarizacijos vektorius
M – įmagnetėjimo vektorius

Maksvelo lygtys:

$$\text{rot}\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} \qquad \text{div}\vec{B} = 0$$

$$\text{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{div}\vec{D} = \rho$$

$$\text{rot}\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \frac{\partial}{\partial t}\left(\varepsilon_0\vec{E} + \vec{P}\right) + \vec{j} \qquad |\mu_0$$

$$\text{rot}\vec{B} - \mu_0\text{rot}\vec{M} = \mu_0\left(\varepsilon_0\frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}\right) + \mu_0\vec{j}$$

$$\begin{cases} \text{rot}\vec{B} = \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} + \mu_0\frac{\partial \vec{P}}{\partial t} + \mu_0\text{rot}\vec{M} + \mu_0\vec{j} \\ \text{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \text{div}\vec{B} = 0 \\ \text{div}\vec{E} = -\frac{1}{\varepsilon_0}\text{div}\vec{P} + \frac{1}{\varepsilon_0}\rho \end{cases}$$

$$\left[\text{div}\vec{D} = \varepsilon_0\text{div}\vec{E} + \varepsilon_0\vec{P} = \rho\right]$$

$$\mu_0\left(\varepsilon_0\frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} + \text{rot}\vec{M} + \vec{j}\right)$$

$j_a = \varepsilon_0\frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} + \text{rot}\vec{M} + \vec{j}$ - apibendrintas srovės tankis

$\varepsilon_0\frac{\partial \vec{E}}{\partial t}$ - slinkties srovės tankis

$\frac{\partial \vec{P}}{\partial t}$ - poliarizacinės srovės tankis

$\text{rot}\vec{M}$ - įmagnetėjimo srovės tankis

$\vec{j}$ - laidumo srovės tankis

$\text{div}\vec{P}$ - poliarizacinio krūvio tankis

Ieškosim elektrinio ir magnetinio vektorių:
$\vec{\pi}_e$ - elektrinis Herzo vektorius $\qquad$ $\vec{\pi}_m$ - magnetinis Herzo vektorius

$$\varphi = -\frac{1}{\varepsilon_0}\mathrm{div}\vec{\pi}_e$$

$$\vec{A} = \mu_0\frac{\partial\vec{\pi}_e}{\partial t} + \frac{1}{\varepsilon_0}\mathrm{rot}\vec{\pi}_m$$

$$\mathrm{div}\vec{A} + \varepsilon_0\mu_0\frac{\partial\varphi}{\partial t} = \mu_0\mathrm{div}\frac{\partial}{\partial t}\vec{\pi}_e + \frac{1}{\varepsilon_0}\mathrm{divrot}\vec{\pi}_m - \frac{1}{\varepsilon_0}\varepsilon_0\mu_0\frac{\partial}{\partial t}\mathrm{div}\vec{\pi}_e = 0$$

Gavom, kad Lorenco sąlyga tenkinama tapatingai.

$$\vec{E} = \frac{1}{\varepsilon_0}\mathrm{graddiv}\vec{\pi}_e - \mu_0\frac{\partial^2\vec{\pi}_e}{\partial t^2} - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\mathrm{rot}\vec{\pi}_m$$

$$\vec{B} = \mu_0\frac{\partial}{\partial t}\mathrm{rot}\vec{\pi}_e + \frac{1}{\varepsilon_0}\mathrm{rotrot}\vec{\pi}_m$$

Pasinaudoję tapatybe $[\mathrm{rotrot}\vec{a} = \mathrm{graddiv}\vec{a} - \Delta\vec{a}]$, gaunam:

$$\vec{E} = \frac{1}{\varepsilon_0}\mathrm{rotrot}\vec{\pi}_e - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\mathrm{rot}\vec{\pi}_m + \frac{1}{\varepsilon_0}\left(\Delta\vec{\pi}_e - \varepsilon_0\mu_0\frac{\partial^2\vec{\pi}_e}{\partial t^2}\right)$$

$$\vec{B} = \frac{1}{\varepsilon_0}\mathrm{rotrot}\vec{\pi}_m + \mu_0\mathrm{rot}\frac{\partial\vec{\pi}_e}{\partial t}$$

Perrašom taip, kad būtų tik gradientai:

$$\vec{E} = \frac{1}{\varepsilon_0}\mathrm{graddiv}\vec{\pi}_e - \mu_0\frac{\partial^2\vec{\pi}_e}{\partial t^2} - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\mathrm{rot}\vec{\pi}_m$$

$$B = \frac{1}{\varepsilon_0}\mathrm{graddiv}\vec{\pi}_m - \mu_0\frac{\partial^2\vec{\pi}_m}{\partial t^2} - \frac{1}{\varepsilon_0}\left(\Delta\vec{\pi}_m - \varepsilon_0\mu_0\frac{\partial^2\vec{\pi}_m}{\partial t^2}\right) + \mu_0\mathrm{rot}\frac{\partial}{\partial t}\vec{\pi}_e$$

Ne visada $\vec{B}$ galima ieškoti kaip gradientų. Tada reikia žinoti ar Laplaso operatorius yra lygus nuliui.

Reikia rasti $\vec{\pi}_e$ ir $\vec{\pi}_m$:

$$\mathrm{rot}\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{j} \qquad \mathrm{div}\vec{B} = 0$$

$$\mathrm{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \mathrm{div}\vec{D} = \rho$$

$$\vec{E} = -\mathrm{grad}\varphi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \mathrm{rot}\vec{A}$$

$$\mathrm{div}\vec{E} = -\frac{1}{\varepsilon_0}\vec{P} + \frac{1}{\varepsilon_0}\rho$$

$$\mathrm{div}\left(\Delta\vec{\pi}_e - \varepsilon_0\mu_0\frac{\partial^2\vec{\pi}_e}{\partial t^2}\right) = -\mathrm{div}\vec{P} + \rho$$

$$\mathrm{rot}\vec{B} = \mu_0\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mathrm{rot}\vec{M} + \frac{\partial\vec{P}}{\partial t} + \vec{j}\right)$$

$$[\mathrm{rotgrad}\varphi = 0]$$

$$-\mu_0\frac{\partial}{\partial t}\left(\Delta\vec{\pi}_e - \varepsilon_0\mu_0\frac{\partial^2\vec{\pi}_e}{\partial t^2}\right) - \frac{1}{\varepsilon_0}\mathrm{rot}\left(\Delta\vec{\pi}_m - \varepsilon_0\mu_0\frac{\partial^2\vec{\pi}_m}{\partial t^2}\right) = \mu_0\frac{\partial\vec{P}}{\partial t} + \mu_0\mathrm{rot}\vec{M} + \mu_0\vec{j}$$

Apribosim: $j = 0$; $\rho = 0$ - kur nėra srovės ir krūvio tankių

$$\Delta\vec{\pi}_e - \varepsilon_0\mu_0 \frac{\partial^2\vec{\pi}_e}{\partial t^2} = -\vec{P} + \mathrm{rot}\vec{Q}$$ $\vec{Q}$ - bet koks vektorius

$$-\frac{\partial}{\partial t}\mathrm{rot}Q - \frac{1}{\varepsilon_0}\mathrm{rot}\left(\Delta\vec{\pi}_m - \varepsilon_0\mu_0 \frac{\partial^2\vec{\pi}_m}{\partial t^2}\right) = \mu_0 \mathrm{rot}\vec{M}$$

$$\Delta\vec{\pi}_m - \varepsilon_0\mu_0 \frac{\partial^2\vec{\pi}_m}{\partial t^2} = -\varepsilon_0\mu_0\vec{M} - \varepsilon_0 \frac{\partial\vec{Q}}{\partial t} - \varepsilon_0 \mathrm{grad}\psi$$

Kadangi: $\vec{E} = -\mathrm{grad}\varphi - \frac{\partial\vec{A}}{\partial t}$ tai:

$\vec{B} = \mathrm{rot}\vec{A}$

$$\Delta\vec{\pi}_e - \varepsilon_0\mu_0 \frac{\partial^2\vec{\pi}_e}{\partial t^2} = -\vec{P} + \vec{B}$$

- Hiugenso principas

$$\Delta\vec{\pi}_m - \varepsilon_0\mu_0 \frac{\partial^2\vec{\pi}_m}{\partial t^2} = -\varepsilon_0\mu_0\vec{M} + \varepsilon_0\vec{E}$$

Vektorių $\vec{P}$ ir $\vec{M}$ fizikinė prasmė

Priklausomybės nuo laiko nėra. Tada:

$$\vec{E} = \frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{\pi}_e + \frac{1}{\varepsilon_0}\Delta\vec{\pi}_e = \frac{1}{\varepsilon_0}\operatorname{graddiv}\vec{\pi}_e$$

$$\vec{B} = \frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{\pi}_m = \frac{1}{\varepsilon_0}\operatorname{graddiv}\vec{\pi}_m - \frac{1}{\varepsilon_0}\Delta\vec{\pi}_m$$

$$\Delta\vec{\pi}_e = -\vec{P} \qquad \vec{\pi}_e = \vec{z}_0\pi_e$$

$$\Delta\vec{\pi}_m = -\varepsilon_0\mu_0\vec{M} \qquad \vec{\pi}_m = \vec{z}_0\pi_m$$

$$\Delta\vec{\pi}_e = \Delta\vec{z}_0\pi_e = \vec{z}_0\Delta\pi_e = -\vec{P} = -\vec{z}_0 p_0$$

$$\Delta\pi_e = -p_0(r)$$

$$\Delta\pi_e = -\delta(r)p_0 \qquad \pi_e = \frac{p_0}{4\pi r}$$

$$\boxed{\vec{\pi}_e = \frac{\vec{z}_0 p_0}{4\pi r} \qquad \vec{\pi}_m = \frac{\vec{z}_0 m_0}{4\pi r}\varepsilon_0\mu_0}$$

Sferinėj koordinačių sistemoj:

$$\vec{z}_0 = \vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta \qquad \operatorname{div}\vec{z}_0\pi_e = \pi_e\operatorname{div}\vec{z}_0 + (\vec{z}_0\operatorname{grad}\pi_e)$$

$$\operatorname{graddiv}\vec{\pi}_e = \operatorname{grad}(\vec{z}_0\operatorname{grad}\pi_e)$$

$$(\vec{z}_0\operatorname{grad}\pi_e) = (\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta)\operatorname{grad}\pi_e = \left(\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta, \operatorname{grad}\frac{p_0}{4\pi r}\right) =$$

$$= \left(\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta, \frac{-\vec{z}_0 p_0}{4\pi r^2}\right) = -\frac{p_0\cos\Theta}{4\pi r^2}$$

$$\vec{E} = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(2\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta\right)$$

$$\vec{E} = \frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{\pi}_e - \vec{z}_0\frac{\delta(r)}{\varepsilon_0}p_0$$

$$\frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{z}_0\frac{p_0}{4\pi r} = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(2\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta\right)$$

Vadinasi dipolio elektrinis laukas yra:

$$\vec{E} = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(2\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta\right) - \frac{\vec{z}_0}{\varepsilon_0}\delta(r)p_0$$

$$\vec{B} = \frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{\pi}_m = \frac{1}{\varepsilon_0}\operatorname{graddiv}\vec{\pi}_m - \frac{1}{\varepsilon_0}\Delta\vec{\pi}_m$$

$$\vec{E} = \frac{1}{\varepsilon_0}\operatorname{graddiv}\vec{\pi}_e = \frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{\pi}_e + \frac{1}{\varepsilon_0}\Delta\vec{\pi}_e$$

$$\vec{\pi}_e = \vec{z}_0\frac{p}{4\pi r} \qquad \vec{\pi}_m = \vec{z}_0\frac{m}{4\pi r}$$

$$\frac{1}{\varepsilon_0}\operatorname{rotrot}\vec{\pi}_e = \frac{1}{\varepsilon_0}\operatorname{graddiv}\vec{\pi}_e = \frac{1}{4\pi\varepsilon_0 r^3}\left(2\vec{r}_0\cos\Theta - \vec{\Theta}_0\sin\Theta\right)$$

$$\Delta\vec{\pi}_m = \vec{z}_0 \Delta \frac{m}{4\pi r} \qquad \Delta\vec{\pi}_e = \vec{z}_0 \Delta \frac{p}{4\pi r}$$

$$\Delta\vec{\pi}_m = \vec{z}_0 \frac{p}{4\pi} \Delta \frac{1}{r} \qquad \Delta\vec{\pi}_e = \vec{z}_0 \frac{m}{4\pi} \Delta \frac{1}{r}$$

$$\Delta\vec{\pi}_m = -\varepsilon_0 \mu_0 \vec{M} \qquad \Delta\vec{\pi}_e = -\varepsilon_0 \vec{P}$$

Laplaso operatorius veikia $\frac{1}{r}$:

$$\Delta\frac{1}{r} = \frac{1}{r^2 \sin\Theta}\left[\sin\Theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{1}{r}\right) + \frac{\partial}{\partial \Theta}\left(\sin\Theta \frac{\partial}{\partial \Theta}\frac{1}{r}\right) + \frac{1}{\sin\Theta}\frac{\partial^2}{\partial \alpha^2}\frac{1}{r}\right]$$

$$\left\langle \frac{\partial}{\partial r}\frac{1}{r} = -\frac{1}{r^2} \right\rangle$$

$$\Delta\frac{1}{r} = \frac{1}{r^2 \sin\Theta} \sin\Theta \frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0 \longrightarrow$$ NETEISINGA, nes negali būt lygu nuliui

$$\Delta\frac{1}{r} dV = \operatorname{divgrad}\frac{1}{r} = \rho$$

$$\int_V \Delta\frac{1}{r} dV = \int_V \operatorname{divgrad}\frac{1}{r} dV = \int_V \rho dV = \oint_S \left(\operatorname{grad}\varphi d\vec{S}\right)$$

$$\operatorname{grad}\varphi = \vec{r}_0 \frac{\partial \varphi}{\partial r} + \vec{\Theta}_0 \frac{1}{r}\frac{\partial \varphi}{\partial \Theta} + \vec{\alpha}_0 \frac{1}{r\sin\Theta}\frac{\partial \varphi}{\partial \alpha} = \vec{r}_0 \frac{\partial}{\partial r}\frac{1}{r} + \vec{\Theta}_0 \frac{1}{r}\frac{\partial}{\partial \Theta}\frac{1}{r} + \vec{\alpha}_0 \frac{1}{\sin\Theta}\frac{\partial}{\partial \alpha}\frac{1}{r} = -\vec{r}_0 \frac{1}{r^2}$$

$$-\oint\left(\vec{r}_0 \frac{1}{r^2}, \vec{r}_0 r^2 \sin\Theta d\Theta d\alpha\right) = \begin{bmatrix} 0 \le \Theta \le \pi & d\vec{S} = \vec{r}_0 r^2 \sin\Theta d\Theta \\ 0 \le \alpha \le 2\pi & (\vec{r}_0 \vec{r}_0) = 1 \end{bmatrix} =$$

$$-\int_0^{2\pi}\int_0^{\pi} \frac{1}{r^2} r^2 \sin\Theta d\Theta d\alpha = -2\pi \int_0^{\pi} \sin\Theta d\Theta = 2\pi \cos\Theta\Big|_0^{\pi} = 2\pi(-1-1) = -4\pi$$

$$\rho = -4\pi\delta(r)$$

$$\int \delta(x) f(x) dx = f(0)$$

$$\vec{\pi}_e = \vec{r}_0 \frac{p}{4\pi r} \qquad \Delta\vec{\pi}_e = -\vec{r}_0 p \delta(r)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left(2\vec{r}_0 \sin\Theta + \vec{\Theta}_0 \sin\Theta\right) + \frac{\vec{z}_0 p \delta(r)}{\varepsilon_0}$$

$$\int_V \delta(r) = 1 \qquad \Delta\vec{\pi}_e = -\varepsilon_0 \vec{P}$$

Tegu turim bet kokį tūrį ir šitam tūryje yra:

$$\vec{z}_0 \sum_{i=1}^{N} p_i \delta(r_i)$$

Paimam vidurkį:

$$\vec{z}_0 \frac{1}{V}\int_V \sum_{i=1}^{N} p_i \delta(r_i) dV = \vec{z}_0 \frac{1}{V}\sum_{i=1}^{N} p_i = \vec{z}_0 \vec{P}$$

$\vec{P}$ - galima fizikinė prasmė – tūrio vieneto vidutinis dipolinis momentas

$$W_E = \frac{1}{2}\left(\vec{E}\vec{D}\right) \qquad W_E = \frac{1}{2}\int_V (\rho\varphi)dV$$

$$\varphi_1 \quad \varphi_2 \quad \rho_1(r_1) \quad \rho_2(r_2) \quad \rho = \rho_1 + \rho_2 \quad \varphi = \varphi_1 + \varphi_2$$

$$W = \frac{1}{2}\int_V (\rho\varphi)dV = \frac{1}{2}\int_V (\rho_1 + \rho_2)(\varphi_1 + \varphi_2)dV =$$

$$= \frac{1}{2}\int_V (\rho_1\varphi_1)dV + \frac{1}{2}\int_V (\rho_2\varphi_2)dV + \frac{1}{2}\int_V (\rho_1\varphi_2)dV + \frac{1}{2}\int_V (\rho_2\varphi_1)dV$$

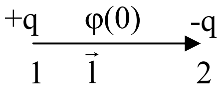
$$W_{12} = \frac{1}{2}\int_V (\rho_1\varphi_2)dV + \frac{1}{2}\int_V (\rho_2\varphi_1)dV$$

$$\varphi_{1,2} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_{1,2}dV(r')}{\left|\vec{r} - \vec{r}_{1,2}\right|}$$

Sąveikos energija:

$$W_{12} = \frac{1}{2}\int_V\int_V \frac{\rho_1(r_1)\rho_2(r_2)dV_1(r_1)}{\left|\vec{r}_1 - \vec{r}_2\right|} + \frac{1}{2}\int_V\int_V \frac{\rho_2(r_2)\rho_1(r_1)dV_1(r_1)dV(r_1)}{\left|\vec{r}_2 - \vec{r}_1\right|}$$

$$W_{12} = \int_V \rho_1\varphi_2 dV_1 = \int_V \rho_2\varphi_1 dV_2$$



$$\varphi(1) \quad \varphi(2) \quad W = q\varphi(1) - q\varphi(2)$$

$$\varphi(1) = \varphi(0) + \left(\frac{\vec{l}}{2}\mathrm{grad}\varphi\right)$$

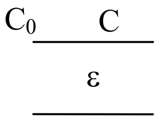
$$\varphi(2) = \varphi(0) - \left(\frac{\vec{l}}{2}\mathrm{grad}\varphi\right)$$

$$W = q\varphi(0) + q\left(\frac{\vec{l}}{2}\mathrm{grad}\varphi\right) - q\varphi(0) + q\left(\frac{\vec{l}}{2}\mathrm{grad}\varphi\right) = \left(q\vec{l}\ \mathrm{grad}\varphi\right) = -\left(\vec{P}\vec{E}\right)$$

$$\Rightarrow \quad W = \pm\left(\vec{P}\vec{E}\right)$$

Kodėl gali būti ir „+“ ir „-“

Tegu turim kondensatorių:



$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \qquad C = \frac{q}{U}$$

$$C > C_0 \qquad W = \frac{1}{2}CU^2 = \frac{1}{2}qU$$

$$q = \mathrm{const}; \qquad W(0) - W(\varepsilon) = \frac{1}{2}q\left(U_0 - U\right) > 0$$

$$U = \mathrm{const}; \qquad W(0) - W(\varepsilon) = \frac{1}{2}U\left(q_0 - q\right) < 0$$

$$W_{12} = \left(\vec{P}_1\vec{E}_2\right) = \left(\vec{P}_2\vec{E}_1\right) = \frac{1}{4\pi\varepsilon_0 r^3}\left[\left(3\vec{r}_0\left(\vec{p}_2\vec{r}_0\right) - p_2\right), \vec{p}_1\right] = \frac{1}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}_1\vec{r}_0\right)\left(\vec{p}_2\vec{r}_0\right) - \left(\vec{p}_1\vec{p}_2\right)\right]$$

Sąveikos energija stipriai priklauso nuo ženklo (plius ar minus). Dipolis yra dviejų lygmenų sistema.

Kokia jėga veikia dipolį patalpintą į išorinį elektrinį lauką

$+q \longrightarrow -q$

$1 \quad 0 \quad 2$

$$\vec{E}(1) = \vec{E}(0) + \left(\frac{\vec{l}}{2}\,\mathrm{grad}\right)\vec{E}$$

$$\vec{E}(2) = \vec{E}(0) - \left(\frac{\vec{l}}{2}\,\mathrm{grad}\right)\vec{E}$$

$$\vec{F} = q\vec{E}(1) - q\vec{E}(2) = q\vec{E}(0) + q\left(\frac{\vec{l}}{2}\,\mathrm{grad}\right)\vec{E} - q\vec{E}(0) - q\left(-\frac{\vec{l}}{2}\,\mathrm{grad}\right)\vec{E} = \left(q\vec{l}\,\mathrm{grad}\right)\vec{E} = \left(\vec{p}\,\mathrm{grad}\right)\vec{E}$$

Dipolio neveikia jokia jėga. Jėga veiktų, jei būtų nevienalytis elektrinis laukas.

$$\vec{F} = \left(\vec{p}\,\mathrm{grad}\right)\vec{E}$$

$$\vec{N} = q\left[\frac{\vec{l}}{2}\vec{E}(1)\right] - q\left[\frac{\vec{l}}{2}\vec{E}(2)\right] = q\left\{\left[\frac{\vec{l}}{2},\vec{E}_0 + \frac{1}{2}\left(\vec{l}\,\mathrm{grad}\right)\vec{E}\right] + \left[\frac{\vec{l}}{2},\vec{E}_0 - \frac{1}{2}\left(\vec{l}\,\mathrm{grad}\right)\vec{E}\right]\right\} = \left[q\vec{l}\vec{E}_0\right] = \left[\vec{p}\vec{E}_0\right]$$

$$\vec{N} = \left[\vec{m}\vec{B}_0\right]$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

Sąveikos energija:

$W_E = \left(\vec{P}\vec{E}\right)$ $\qquad$ $W_M = \left(\vec{M}\vec{B}\right)$

$\vec{F}_d = \left(\vec{p}\,\mathrm{grad}\right)\vec{E}$ $\qquad$ $\vec{F}_m = \left(\vec{m}\,\mathrm{grad}\right)\vec{B}$

$$\vec{P} = (\varepsilon - \varepsilon_0)\vec{E}$$

$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} = \frac{1}{\mu}\vec{B};$ $\qquad$ $\vec{M} = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\vec{B}$

$\vec{F}_d = (\varepsilon - \varepsilon_0)\left(\vec{E}\,\mathrm{grad}\right)\vec{E};$ $\qquad$ $\vec{F}_m = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\left(\vec{B}\,\mathrm{grad}\right)\vec{B}$

Pasinaudojam tapatybe:

$$\mathrm{grad}\left(\vec{a}\vec{b}\right) = \left[\vec{a}\,\mathrm{rot}\vec{b}\right] + \left[\vec{b}\,\mathrm{rot}\vec{a}\right] + \left(\vec{b}\,\mathrm{grad}\right)\vec{a} + \left(\vec{a}\,\mathrm{grad}\right)\vec{b}$$

$\vec{a} = \vec{b} \qquad \mathrm{rot}\vec{a} = \mathrm{rot}\vec{b} = 0$

$$\mathrm{grad}\left(\vec{a}\vec{a}\right) = 2\left(\vec{a}\,\mathrm{grad}\right)\vec{a}$$

$$\vec{F}_d = \frac{1}{2}(\varepsilon - \varepsilon_0)\mathrm{grad}\left(\vec{E}\vec{E}\right) = \frac{1}{2}(\varepsilon - \varepsilon_0)\mathrm{grad}\left|\vec{E}\right|^2$$

$$\vec{F}_m = \frac{1}{2}\left(\frac{\mu - \mu_0}{\mu\mu_0}\right)\mathrm{grad}\left(\vec{B}\vec{B}\right) = \frac{1}{2}\left(\frac{\mu - \mu_0}{\mu\mu_0}\right)\mathrm{grad}\left|\vec{B}\right|^2$$

Nuostovusis elektromagnetinis laukas

Tai elektromagnetinis laukas, kuriame nėra priklausomybės nuo laiko.

$$\mathrm{rot}\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} = \vec{j}; \qquad \frac{\partial}{\partial t} = 0$$

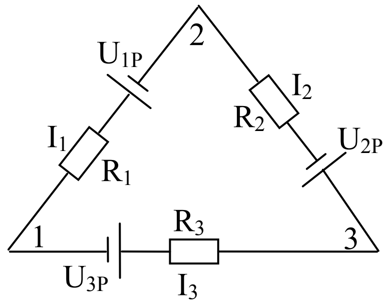
$$\mathrm{rot}\vec{E} = \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{D} = \varepsilon \vec{E}$$

$$\mathrm{div}\vec{B} = 0 \qquad \vec{B} = \mu \vec{H}$$

$$\mathrm{div}\vec{D} = \rho \qquad \vec{j} = \sigma \vec{E}$$

$$\begin{cases} \mathrm{rot}\vec{H} = \vec{j} \\ \mathrm{rot}\vec{E} = 0 \\ \mathrm{div}\vec{B} = 0 \\ \mathrm{div}\vec{D} = \rho \end{cases}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{j} = 0 \qquad \mathrm{div}\vec{j} = 0$$



$$\int_V \mathrm{div}\vec{j}\,dV = \int_S \left(\vec{j}\,d\vec{S}\right) = \int_{S_1} \left(\vec{j}_1 d\vec{S}_1\right) + \int_{S_2} \left(\vec{j}_2 d\vec{S}_2\right) + \int_{S_3} \left(\vec{j}_3 d\vec{S}_3\right)$$

$I_1 + I_2 + I_3 = 0$ → I Kirchhofo dėsnis

$$\vec{E} = -\mathrm{grad}\varphi + \vec{E}_P$$

$$\mathrm{rot}\vec{E} = 0 \qquad \vec{j} = \sigma \vec{E}$$

$$I_1 R_1 = \underbrace{-\int_1^2 \left(\mathrm{grad}\varphi\, d\vec{l}\right)}_{U_{12}} + \underbrace{\int_1^2 \left(\vec{E}_P d\vec{l}\right)}_{U_{1P}}$$

$$I_1 R_1 + I_2 R_2 + I_3 R_3 = U_{12} + U_{1P} + U_{23} + U_{2P} + U_{31} + U_{3P}$$

$$U_{12} + U_{23} + U_{31} = 0$$

$I_1 R_1 + I_2 R_2 + I_3 R_3 = U_{1P} + U_{2P} + U_{3P}$ ⟶ II Kirchhofo dėsnis

Kraštinės sąlygos:

$$\mathrm{div}\vec{j} = 0 \qquad B_{1n} = B_{2n} \qquad \vec{j} = \sigma \vec{E}$$

$$\mathrm{div}\vec{B} = 0 \qquad j_{1n} = j_{2n} \qquad \vec{E}_{1\tau} = \vec{E}_{2\tau}$$

$$\int_S \left(\vec{j}_1 - \vec{j}_2, d\vec{S}\right) = 0$$

$$\frac{\vec{j}_{1\tau}}{\sigma_1} = \vec{E}_{1\tau} = \frac{\vec{j}_{2\tau}}{\sigma_2} = \vec{E}_{2\tau}$$

$$\frac{1}{\sigma_1}\vec{j}_{1\tau} = \frac{1}{\sigma_2}\vec{j}_{2\tau}$$

Pvz.: $\sigma = \infty$ - jei viena aplinka superlaidi, o kita ne, bus tik normalinės srovės komponentės.

$$P = \int_V \left(\vec{j}\vec{E}\right) dV = \underbrace{-\int_V \left(\vec{j}\,\mathrm{grad}\varphi\right) dV}_{=0} + \int_V \left(\vec{j}\vec{E}_P\right) dV$$

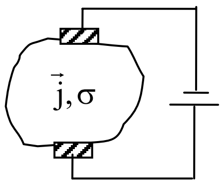
$$\left[\mathrm{div}(\vec{a}\cdot f) = f\,\mathrm{div}\vec{a} + (\vec{a}\,\mathrm{grad} f)\right]$$

$$\mathrm{div}(\vec{j}\varphi) = \varphi\underbrace{\mathrm{div}\vec{j}}_{=0} + (\vec{j}\,\mathrm{grad}\varphi)$$

$$\int\limits_V (\vec{j}\,\mathrm{grad}\varphi)dV = \int\limits_V \mathrm{div}(\vec{j}\varphi)dV = \int\limits_S (\vec{j}\varphi d\vec{S}) = \int\limits_S \varphi(\vec{j}d\vec{S}) = 0$$

$$P = \int\limits_V (\vec{j}d\vec{E}_P)dV$$

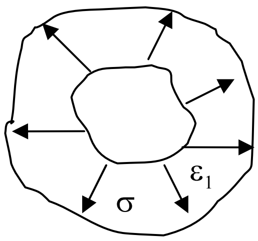
Tegu turim laidžią aplinką, kur norim sužinot kaip pasiskirstęs srovės tankis:



$$\vec{j} = \sigma\vec{E} = -\sigma\,\mathrm{grad}\varphi$$

$$\mathrm{div}\vec{j} = -\sigma\,\mathrm{div}\,\mathrm{grad}\varphi = 0$$

$$\Delta\varphi = \mathrm{div}\,\mathrm{grad}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$



$$\vec{D} = \varepsilon\vec{E} \qquad \vec{j} = \sigma\vec{E}$$

$$\vec{D}_{1n} = \vec{D}_{2n} \qquad \frac{1}{\varepsilon_1}\vec{D}_{1\tau} = \frac{1}{\varepsilon_2}\vec{D}_{2\tau}$$

$$\vec{j}_{1n} = \vec{j}_{2n} \qquad \frac{1}{\sigma_1}\vec{E}_{1\tau} = \frac{1}{\sigma_2}\vec{E}_{2\tau}$$

$$\vec{D} = -\varepsilon\vec{E} = -\varepsilon\,\mathrm{grad}\varphi \qquad \vec{j} = \sigma\vec{E} = -\sigma\,\mathrm{grad}\vec{E}$$

$$q = \int\limits_S (\vec{D}d\vec{S}) \qquad I = \int\limits_S (\vec{j}d\vec{S})$$

$$\Delta\varphi = 0$$

$$C = \frac{q}{\varphi_1 - \varphi_2}; \qquad \frac{1}{R} = \frac{I}{\varphi_1 - \varphi_2}$$

Sūkurinės elektromagnetinės bangos

$$\vec{E}(x,y,z,t) = \left\{\vec{x}_0 U(x,y) - \vec{z}_0 \frac{i}{k}\frac{\partial U(x,y)}{\partial x}\right\} \cdot e^{i(wt-kz)}$$

$$k = w\sqrt{\varepsilon\mu} \qquad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad \mathrm{rot}\vec{E} = \text{-}iw\mu\vec{H}$$

$$\mathrm{rot}\vec{E} = \begin{vmatrix} \vec{x}_0 & \vec{y}_0 & \vec{z}_0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & E_z \end{vmatrix} = \vec{x}_0 \frac{\partial E_z}{\partial y} + \vec{y}_0\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) - \vec{z}_0 \frac{\partial E_x}{\partial y} = -iw\mu\left(\vec{x}_0 H_x + \vec{y}_0 H_y + \vec{z}_0 H_z\right)$$

$$E_x = U(x,y) \qquad E_z = -\frac{i}{k}\frac{\partial U(x,y)}{\partial x}$$

$$-iw\mu\vec{H} = -\vec{x}_0 \frac{i}{k}\frac{\partial^2 U}{\partial x \partial y} + \vec{y}_0\left(-ikU + \frac{i}{k}\frac{\partial^2 U}{\partial x^2}\right) - \vec{z}_0 \frac{\partial U}{\partial y}$$

$$\vec{H} = \frac{i}{w\mu}\left\{\vec{x}_0\left(-\frac{i}{k}\frac{\partial^2 U}{\partial x \partial y}\right) + \vec{y}_0\left(-ikU + \frac{i}{k}\frac{\partial^2 U}{\partial x^2}\right) - \vec{z}_0 \frac{\partial U}{\partial y}\right\} \cdot e^{i(wt-kz)}$$

$$\mathrm{rot}\vec{H} = iw\varepsilon\vec{E} = \vec{x}_0\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \vec{y}_0\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \vec{z}_0\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\vec{E} = \frac{1}{iw\varepsilon}\frac{i}{w\mu}\left\{\vec{x}_0\left(-\frac{\partial^2 U}{\partial y^2} + k^2 U - \frac{k}{k}\frac{\partial^2 U}{\partial x^2}\right) + \vec{y}_0\left(-\frac{k}{k}\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 U}{\partial x \partial y}\right) + \right.$$

$$\left. + \vec{z}_0\left(-ik\frac{\partial U}{\partial x} + \frac{i}{k}\frac{\partial}{\partial x}\frac{\partial^2 U}{\partial x^2} + \frac{i}{k}\frac{\partial}{\partial x}\frac{\partial^2 U}{\partial y^2}\right)\right\} \cdot e^{i(wt-kz)}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$$\left.\begin{aligned} &\mathrm{div}\vec{B} = 0 \\ &\mathrm{div}\vec{D} = \varepsilon\,\mathrm{div}\vec{E} = 0 \\ &\mathrm{div}\vec{B} = \mu\,\mathrm{div}\vec{H} = 0 \end{aligned}\right\} \quad \text{- Šitų lygčių netikrinam}$$

$$\mathrm{rot}\vec{E} = -iw\mu\vec{H}$$

$$\mathrm{rot}\vec{H} = iw\varepsilon\vec{E}$$

Jei tenkinamos šitos lygtys, tai tapatingai bus tenkinamos ir kitos (kurių netikrinom).

Rasim <u>kompleksinį Pointingo vektorių:</u>

$$\vec{\pi} = \frac{1}{2}\left[\vec{E}\,\overset{*}{\vec{H}}\right] = \frac{1}{2}\begin{vmatrix} \vec{x}_0 & \vec{y}_0 & \vec{z}_0 \\ E_x & E_y & E_z \\ \overset{*}{H}_x & \overset{*}{H}_y & \overset{*}{H}_z \end{vmatrix} =$$

$$= \frac{1}{2}\left\{\vec{x}_0\left(E_y \overset{*}{H}_z - E_z \overset{*}{H}_y\right) + \vec{y}_0\left(E_z \overset{*}{H}_x - E_x \overset{*}{H}_z\right) + \vec{z}_0\left(E_x \overset{*}{H}_y - E_y \overset{*}{H}_x\right)\right\}$$

$$\vec{\pi}=\frac{1}{2w\mu}\left\{\vec{x}_0 i\left(U\frac{\partial U}{\partial x}-\frac{1}{k^2}\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2}\right)+\vec{y}_0 i\left(-\frac{1}{k^2}\frac{\partial^2 U}{\partial x\partial y}\frac{\partial U}{\partial x}+U\frac{\partial U}{\partial y}\right)+\vec{z}_0\left(kU^2-\frac{1}{k}U\frac{\partial^2 U}{\partial x^2}\right)\right\}$$

$$e^{i(wt-kz)}=e^{i(wt-(\vec{z}_0 k,\vec{z}_0 z))}$$

k vektorius nukreiptas z ašies kryptimi. Norint matyti sūkurius, viską reikia užrašyti realiom dalim:

$$\vec{E}=\vec{x}_0 U\cos(wt-kz)+\vec{z}_0\frac{1}{k}\frac{\partial U}{\partial x}\sin(wt-kz)$$

$$\vec{H}=\frac{1}{w\mu}\left\{\vec{x}_0\frac{1}{k}\frac{\partial^2 U}{\partial x\partial y}\cos(wt-kz)+\vec{y}_0\left(kU-\frac{1}{k}\frac{\partial^2 U}{\partial x^2}\right)\cos(wt-kz)+\vec{z}_0\frac{\partial U}{\partial y}\sin(wt-kz)\right\}$$

$$\vec{\pi}=\left[\vec{E}\vec{H}\right]=\frac{1}{w\mu}\left\{-\vec{x}_0\frac{1}{k}\frac{\partial U}{\partial x}\left(kU-\frac{1}{k}\frac{\partial^2 U}{\partial x^2}\right)\sin(wt-kz)\cos(wt-kz)+\right.$$

$$+\vec{y}_0\left[\frac{1}{k}\frac{\partial U}{\partial x}\frac{1}{k}\frac{\partial^2 U}{\partial x\partial y}\sin(wt-kz)\cos(wt-kz)-U\frac{\partial U}{\partial y}\sin(wt-kz)\cos(wt-kz)\right]+$$

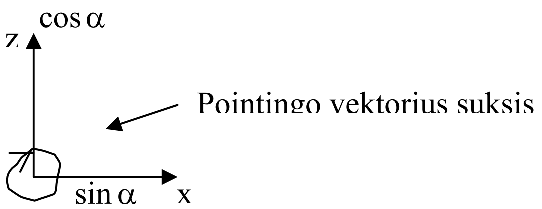
$$\left.+\vec{z}_0 U\left(kU-\frac{1}{k}\frac{\partial^2 U}{\partial x^2}\cos^2(wt-kz)\right)\right\}$$

Pasinaudojam šitom tapatybėm:

$$\left[\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x;\qquad \sin x\cdot\cos x=\frac{1}{2}\sin 2x\right]$$

$$\vec{\pi}=\frac{1}{2w\mu}\left\{-\vec{x}_0\frac{1}{k}\frac{\partial U}{\partial x}\left(kU-\frac{1}{k}\frac{\partial^2 U}{\partial x^2}\right)\sin[2(wt-kz)]+\vec{y}_0\left(\frac{1}{k}\frac{\partial U}{\partial x}\frac{1}{k}\frac{\partial^2 U}{\partial x\partial y}-U\frac{\partial U}{\partial y}\right)\sin[2(wt-kz)]+\right.$$

$$\left.+\vec{z}_0 U\left(kU-\frac{1}{k}\frac{\partial^2 U}{\partial x^2}\right)\cos[2(wt-kz)]\right\}+\vec{z}_0\frac{1}{2w\mu}U\left(kU-\frac{1}{k}\frac{\partial^2 U}{\partial x^2}\right)$$



$$U=x^2-y^2;\qquad \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=2-2=0$$

$$\frac{\partial^2 U}{\partial x\partial y}=0;\quad \frac{\partial U}{\partial x}=2x;\quad \frac{\partial U}{\partial y}=-2y;\quad \frac{\partial^2 U}{\partial x^2}=2$$

$$\vec{E}=\vec{x}_0(x^2-y^2)\cos(wt-kz)+\vec{z}_0\frac{2x}{k}\sin(wt-kz)$$

$$\vec{H}=\frac{1}{w\mu}(\vec{y}_0 k(x^2-y^2)\cos(wt-kz)-\frac{2}{k}\sin(wt-kz)-\vec{z}_0 2y\sin(wt-kz))$$

$$\vec{\pi}=\frac{1}{2w\mu}\left\{\left(-\vec{x}_0 2x\left(x^2-y^2-\frac{2}{k^2}\right)\right)\sin[2(wt-kz)]-\vec{y}_0 2y(x^2-y^2)\sin[2(wt-kz)]+\right.$$

$$\left.+\vec{z}_0(x^2-y^2)\left(x^2-y^2-\frac{2}{k^2}\right)\cos[2(wt-kz)]\right\}+\vec{z}_0\frac{1}{2w\mu}(x^2-y^2)k\left(x^2-y^2-\frac{2}{k^2}\right)$$

Harmoniniai virpesiai ir neapibrėžtumo principas

Furje transformacija:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{iwt} dw$$

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt; \qquad \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

$$I = \int_{-\infty}^{\infty} e^{-\pi x^2} dx \qquad \text{- Gauso integralas (pasiskirstymas)}$$

$$\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right)^2 = \int_{-\infty}^{\infty} e^{-\pi y^2} dy \int_{-\infty}^{\infty} e^{-\pi x^2} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi (x^2 + y^2)} dx dy$$

$$\rho = \sqrt{x^2 + y^2}; \qquad dxdy = \rho d\rho d\alpha$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right)^2 = \int_0^{\infty} \int_0^{2\pi} \rho e^{-\pi \rho^2} d\rho d\alpha = 2\pi \int_0^{\infty} \rho e^{-\pi \rho^2} d\rho = \int_0^{\infty} e^{-\pi \rho^2} d\left(\pi \rho^2\right) = \int_0^{\infty} e^{-u} du = -e^{-u}\Big|_0^{\infty} = 1$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1; \qquad f(t) = e^{-\pi \left(\frac{t}{\Delta t} \right)^2}$$

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\pi \left(\frac{t}{\Delta t} \right)^2} \cdot e^{-iwt} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\pi \left(\frac{t}{\Delta t} \right)^2 + iwt \right]} dt$$

$$w = 2\pi\nu \qquad \text{- ciklinis daznis}$$

$$\pi \left(\frac{t}{\Delta t} \right)^2 + i2\pi\nu t = \pi \left\{ \left(\frac{t}{\Delta t} + i\nu\Delta t \right)^2 + \nu^2 \Delta t^2 \right\}$$

$$\pi \nu^2 \Delta t^2 = \frac{w^2 \Delta t^2}{4\pi}; \qquad \text{Pakeitimas: } \frac{t}{\Delta t} + i\nu\Delta t = x; \qquad dt = \Delta t dx$$

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\pi \left(\frac{t}{\Delta t} + i\nu\Delta t \right)^2} \cdot e^{-\frac{w^2 \Delta t^2}{4\pi}} dt$$

$$\int_{-\infty}^{\infty} e^{-\pi \left(\frac{t}{\Delta t} + i\nu\Delta t \right)^2} dt = \Delta t$$

$$F(w) = \frac{\Delta t}{\sqrt{2\pi}} e^{-\frac{w^2 \Delta t^2}{4\pi}}$$

$$\text{Tegu:} \qquad f(t) = A e^{-\pi \left(\frac{t}{\Delta t} \right)^2} \cos w_0 t = A \frac{1}{2} e^{-\pi \left(\frac{t}{\Delta t} \right)^2} \left\{ e^{iw_0 t} + e^{-iw_0 t} \right\}; \qquad w_0 \text{ - fiksuotas dydis}$$

$$F(w)=\frac{A}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{-\pi\frac{t}{\Delta t}}\cdot e^{-\pi(w-w_0)}dt+\int_{-\infty}^{\infty} e^{-\pi\left(\frac{t}{\Delta t}\right)^2}\cdot e^{-\pi(w+w_0)}dt\right\}=$$

$$=\frac{A}{2\sqrt{2\pi}}\Delta t\left\{e^{-\frac{(w-w_0)^2\Delta t^2}{4\pi}}+e^{-\frac{(w+w_0)^2\Delta t^2}{4\pi}}\right\}$$

$$f(t)=Ae^{-\pi\left(\frac{t}{\Delta t}\right)^2}\cdot\cos\left(w_0\Delta t\frac{t}{\Delta t}\right)$$

$$\frac{t}{\Delta t}<<1 \quad\Rightarrow\quad e\sim 1$$

$$w_0\Delta t\frac{t}{\Delta t}>2\pi n$$

$w_0\Delta t>>1 \quad\Rightarrow\quad$ Periodinis virpejimas

Vektorinis ir skaliarinis potencialas kaip 4-matis vektorius

$\{x, y, z, v_0 t\}$ - Keturmatis vektorius

Visiem stebėtojam atstumas turi būti tas pats (pereinant vektoriui iš vienos koord. sist. į kitą).

T.y.: $\sqrt{x^2+y^2+z^2} = S;$ $\qquad x^2+y^2+z^2 = S^2$

$$|x, y, z| \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix} = |x, y, z| \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix} = x^2 + y^2 + z^2; \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \text{ - Metrinis tenzorius}$$

$$|x_1 y_1 z_1| \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Keturmatėj erdvėj metrinis tenzorius - $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

$$|x, y, z, v_0 t| \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} x \\ y \\ z \\ v_0 t \end{vmatrix} = x^2 + y^2 + z^2 - v_0^2 t^2 = S^2$$

v_0 – šitas greitis priklauso kokioj aplinkoj sklinda bangos (bangos sklidimo greitis, nieko bendro su šviesos greičiu)

Euklidinis tenzorius - $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ $\qquad$ Tvistorinė erdvė - $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

Maksvelo lygtys laisvoj erdvėj: $\begin{cases} \vec{D} = \varepsilon \vec{E} \qquad \mathrm{rot}\vec{H} = \varepsilon \dfrac{\partial \vec{E}}{\partial t} \qquad \varepsilon \mathrm{div}\vec{E} = 0 \\ \vec{B} = \mu \vec{H} \qquad \mathrm{rot}\vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t} \qquad \mu \mathrm{div}\vec{H} = 0 \end{cases}$

$$\vec{E}(x, y, z, t) = \vec{E}(f(x, y, z, t))$$

$$\vec{H}(x, y, z, t) = \vec{H}(f(x, y, z, t))$$

$$\vec{E} = \vec{E}_0 e^{i(wt - (\vec{k}\vec{z}))} \qquad f = wt - xk_x - yk_y - zk_z$$

$$\mathrm{rot}\vec{E}\Big|_x = \vec{x}_0 \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) = \frac{\partial E_z}{\partial f} \frac{\partial f}{\partial y} - \frac{\partial E_y}{\partial f} \frac{\partial f}{\partial z}$$

$$\left[\vec{A}\vec{B}\right]_x = \vec{x}_0 \left(A_y B_z - B_y A_z \right)$$

$$\mathrm{rot}\vec{H} = \left[\mathrm{grad} f, \frac{\partial \vec{H}}{\partial f} \right]$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial f}{\partial t} \frac{\partial \vec{E}}{\partial f}; \qquad \mathrm{div}\vec{E} = \left[\mathrm{grad} f, \frac{\partial \vec{E}}{\partial f} \right]$$

$$\left.\begin{aligned}&\left[\text{gradf},\frac{\partial\vec{H}}{\partial f}\right]=\varepsilon\frac{\partial f}{\partial t}\frac{\partial\vec{E}}{\partial f}\\&\left[\text{gradf},\frac{\partial\vec{E}}{\partial f}\right]=-\mu\frac{\partial f}{\partial t}\frac{\partial\vec{H}}{\partial f}\\&\left(\text{gradf},\frac{\partial\vec{E}}{\partial f}\right)=0\\&\left(\text{gradf},\frac{\partial\vec{H}}{\partial f}\right)=0\end{aligned}\right\}\quad\text{- Taip virsta Maksvelo lygtys}$$

$$\left[\vec{a}\left[\vec{b}\vec{c}\right]\right]=\vec{b}(\vec{a}\vec{c})-\vec{c}(\vec{a}\vec{b})$$

$$\left[\text{gradf},\left[\text{gradf},\frac{\partial\vec{H}}{\partial f}\right]\right]=\varepsilon\frac{\partial f}{\partial t}\left[\text{gradf},\frac{\partial\vec{E}}{\partial f}\right]$$

$$\left[\text{gradf},\left[\text{gradf},\frac{\partial\vec{E}}{\partial f}\right]\right]=-\mu\frac{\partial f}{\partial t}\left[\text{gradf},\frac{\partial\vec{H}}{\partial f}\right]$$

$$\left\{\text{gradf},\left[\text{gradf},\frac{\partial\vec{H}}{\partial f}\right]\right\}=-\varepsilon\mu\left(\frac{\partial f}{\partial t}\right)^2\frac{\partial\vec{H}}{\partial f}$$

Pritaikom tapatybe:

$$\text{gradf}\left(\text{gradf},\frac{\partial\vec{H}}{\partial f}\right)-\frac{\partial\vec{H}}{\partial f}(\text{gradf},\text{gradf})=-\varepsilon\mu\left(\frac{\partial f}{\partial t}\right)^2\frac{\partial\vec{H}}{\partial f}$$

$$\left[\left(\text{gradf},\frac{\partial\vec{H}}{\partial f}\right)=0=\text{div}\vec{H}\right]$$

$$-(\text{gradf},\text{gradf})\frac{\partial\vec{H}}{\partial f}=-\varepsilon\mu\left(\frac{\partial f}{\partial t}\right)^2\frac{\partial\vec{H}}{\partial f}$$

$$-\left(\frac{\partial f}{\partial x}\right)^2-\left(\frac{\partial f}{\partial y}\right)^2-\left(\frac{\partial f}{\partial z}\right)^2+\varepsilon\mu\left(\frac{\partial f}{\partial t}\right)^2=0;\qquad\left[\varepsilon\mu=\frac{1}{c^2}\right]$$

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2-\left(\frac{\partial f}{\partial(ct)}\right)^2=0$$

$$x^2+y^2+z^2-(ct)^2=S^2$$

$$f=wt-xk_x-yk_y-zk_z$$

$$\frac{\partial f}{\partial x}=k_x;\quad\frac{\partial f}{\partial y}=k_y;\quad\frac{\partial f}{\partial z}=k_z;\quad\frac{\partial f}{\partial t}=w$$

$$k_x^2+k_y^2+k_z^2-\left(\frac{w}{c}\right)^2=0$$

4 - matis vektorius $\rightarrow\ \{k_x,k_y,k_z,k\}\qquad k=\frac{w}{c}$

Vektorius ir kovektorius

$$\vec{x}' = \{x'_1, x'_2, x'_3, x'_4\} \qquad \varphi(x'_1, x'_2, x'_3, x'_4)$$

$$\mathrm{grad}\varphi = \left\{\frac{\partial\varphi}{\partial x'_1}, \frac{\partial\varphi}{\partial x'_2}, \frac{\partial\varphi}{\partial x'_3}, \frac{\partial\varphi}{\partial x'_4}\right\}$$

Atliekam Lorenco transformacija :

$$\vec{x} = \{x_1, x_2, x_3, x_4\}$$

$$x_i = \underbrace{a_{i1}x'_1 + a_{i2}x'_2 + a_{i3}x'_3 + a_{i4}x'_4}_{\text{vektorius}}$$

$$\vec{x} = \hat{A}\vec{x}' \qquad x'_j = a^{-1}_{j1}x_1 + a^{-1}_{j2}x_2 + a^{-1}_{j3}x_3 + a^{-1}_{j4}x_4$$

$$\hat{A}^{-1}\vec{x} = \hat{A}^{-1}\hat{A}\vec{x}' \qquad \frac{\partial\varphi}{\partial x_i} = \sum_{k=1}^{4}\frac{\partial\varphi}{\partial x_k}\frac{\partial x'_k}{\partial x_i} = \underbrace{a^{-1}_{1i}\frac{\partial\varphi}{\partial x'_1} + a^{-1}_{2i}\frac{\partial\varphi}{\partial x'_2} + a^{-1}_{3i}\frac{\partial\varphi}{\partial x'_3} + a^{-1}_{4i}\frac{\partial\varphi}{\partial x'_4}}_{\text{kovektorius}}$$

$$\vec{x}' = \hat{A}^{-1}\vec{x}$$

$$\left(A^{-1}\right)^T = \hat{A}^{-1} \qquad \text{- ortogonali transformacija}$$

$$\vec{E} = -\mathrm{grad}\varphi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \mathrm{rot}\vec{A}$$

Lorenco kalibravimo salyga :

$$\mathrm{div}\vec{A} + \varepsilon_0\mu_0\frac{\partial\varphi}{\partial t} = \mathrm{div}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z + \frac{\partial}{\partial(ct)}\left(\frac{\varphi}{c}\right)$$

$$\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial(ct)}\end{vmatrix} \cdot \begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} \cdot \begin{vmatrix}A_x\\ A_y\\ A_z\\ \frac{\varphi}{c}\end{vmatrix} = 0 \qquad \left\{A_x, A_y, A_z, \frac{\varphi}{c}\right\}$$

$$x = \frac{1}{\sqrt{1-v^2/c^2}}x' + \frac{v}{c}\frac{ct'}{\sqrt{1-v^2/c^2}} \qquad A_x = \frac{1}{\sqrt{1-v^2/c^2}}A'_x + \frac{v}{c}\frac{\frac{\varphi'}{c}}{\sqrt{1-v^2/c^2}}$$

$$ct = \frac{v}{c}\frac{1}{\sqrt{1-v^2/c^2}}x' + \frac{1}{\sqrt{1-v^2/c^2}}ct' \qquad \frac{\varphi}{c} = \frac{v}{c}\frac{1}{\sqrt{1-v^2/c^2}}A'_x + \frac{1}{\sqrt{1-v^2/c^2}}\frac{\varphi'}{c}$$

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{\partial}{\partial x'} - \frac{v}{c}\frac{1}{\sqrt{1-v^2/c^2}}\frac{\partial}{\partial ct'} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

$$\frac{\partial}{\partial ct} = -\frac{v}{c}\frac{1}{\sqrt{1-v^2/c^2}}\frac{\partial}{\partial x'} + \frac{1}{\sqrt{1-v^2/c^2}}\frac{\partial}{\partial ct'} \qquad \frac{\partial A_x}{\partial x} + \frac{\partial}{\partial ct}\frac{\varphi}{c} = \frac{\partial A_x}{\partial x'} + \frac{\partial}{\partial ct'}\frac{\varphi'}{c}$$